

# Probing $CPT$ violation in meson mixing by non-cyclic phase

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The presence of non-cyclic phases is revealed in the time evolution of mixed meson systems. Such phases are related to the parameter  $z$  describing the  $CPT$  violation; moreover, a non zero phase difference between particle and antiparticle arises only in presence of  $CPT$  symmetry breaking. Thus, a completely new test for the  $CPT$  invariance can be provided by the study of such phases in mixed mesons. Systems which are particularly interesting for such an analysis are the  $B_s^0 - \bar{B}_s^0$  and the  $K^0 - \bar{K}^0$  ones. In order to introduce non-cyclic phases, some aspects of the formalism describing the mixed neutral mesons are analyzed. Since the effective Hamiltonian of systems like  $K^0 - \bar{K}^0$ ,  $B^0 - \bar{B}^0$ ,  $B_s^0 - \bar{B}_s^0$ ,  $D^0 - \bar{D}^0$  is non-Hermitian and non-normal, it is necessary to diagonalize it by utilizing the rules of non-Hermitian quantum mechanics.

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## I. INTRODUCTION

Quantum mixing of particles is among the most intriguing topics in subnuclear physics. The theoretical aspects of this phenomenon have been analyzed thoroughly in the contexts of quantum mechanics (QM) [1]–[4] and of quantum field theory (QFT) [5]–[15] where modifications to the QM oscillation formulas have been obtained. The field-theoretical corrections, due to the nonperturbative vacuum structure associated with particle mixing, may be as large as 5 – 20% for strongly mixed systems, such as  $\omega - \phi$  or for  $\eta - \eta'$ , [7]. On the contrary in meson systems as  $K^0$ ,  $B_d^0$ ,  $B_s^0$  and  $D^0$  and in the fermion sector these corrections are negligible. Then, although the QFT analysis discloses features which cannot be ignored (see for example Refs.[12]–[15]), nevertheless a correct phenomenological description of systems as  $B^0 - \bar{B}^0$  can be also dealt with in the context of QM, neglecting the nonperturbative field-theoretical effects.

The analysis of mixed meson systems has played a crucial role in the phenomenology. Indeed the mixing of  $K^0 - \bar{K}^0$  provided the first evidence of  $CP$  violation in weak interactions [16] and the  $B^0 - \bar{B}^0$  mixing is used to determine experimentally the precise profile of CKM unitarity triangle [17], [18]. Moreover, particle mixing offers the possibility to investigate new physics beyond the Standard Model of elementary particle physics, in particular allows to test the validity of the  $CPT$  symmetry which is supposed to be an exact symmetry. Up to now all possible tests are consistent with no  $CPT$  violation [19]; however, new and much more precise measurements are expected in the next generation of experiments at LHC, where  $B_s^0$  and  $B_d^0$  mesons will be abundantly produced and where the very high time resolution of order of 40fs of the detectors ATLAS and CMS will permit to track precisely the time evolution of the  $B$  particles.

On the other hand, in recent years great attention has been devoted to the study of geometric phases [20]–[35] which appear in the evolution of many physical systems. Berry like phases and non-cyclic invariants associated to neutrino oscillations (see for example [36]–[39] and references therein) and to non-hermitian systems (see for example [24], [40]–[43] and references therein) have been also studied extensively.

In the present paper, it is shown that these most interesting issues are intimately bound together in such a way that the non-cyclic phases for mixed meson systems appear to provide a new instrument to test the  $CPT$  symmetry. It is shown that phases such as the Mukunda–Simon ones [24], appearing as observable characterization of the mixed mesons evolution, are related to the parameter denoting the  $CPT$  violation. In particular, the presence of non-trivial Mukunda–Simon phases and of a phase difference between particle and antiparticle indicates unequivocally the  $CPT$  symmetry breaking in mixed boson systems. Furthermore, it is shown that the non-cyclic phases can be useful also to analyze the  $CP$  violation. An especially interesting system for studying the geometric phases is the  $B_s^0 - \bar{B}_s^0$  one because a lot of particle-antiparticle oscillations occur within its lifetime. Thus, the next experiments on the  $B_s^0$  mesons might open new horizons to be explored in future research.

In Appendix C, the Aharonov–Anandan invariants [23] for mixed mesons are presented and their relation with the parameter describing  $CP$  violation is shown.

In order to study the non-cyclic phases, some of the features of the formalism depicting the evolution of mixed neutral mesons in QM have been analyzed. Since in the Wigner–Weisskopf approximation [44] the effective Hamiltonian describing such systems is non-Hermitian and non-normal, to diagonalize it the rules of non-Hermitian quantum mechanics have to be used. In this work, the biorthonormal basis formalism [45]–[48] will be used.

The structure of the paper is the following: in Section II the effective Hamiltonian  $\mathcal{H}$  of mixed meson systems is diagonalized by a complete biorthonormal set of states. The Mukunda–Simon phases for mixed mesons, their

connections with  $CPT$  and  $CP$  violations and the analysis of such phases for  $B_s$  mesons are presented in Section III. Section IV is devoted to the conclusions.

Useful expressions of the states  $|M^0(t)\rangle$  and  $|\bar{M}^0(t)\rangle$  are reported in Appendix A. In Appendix B, the asymmetries describing the  $T$  and  $CPT$  violations are computed using the biorthonormal basis formalism. They coincide with the corresponding ones obtained by employing the states usually adopted in the literature. The Aharonov–Anandan invariants are studied in Appendix C.

## II. MESON MIXING AND BIORTHONORMAL BASIS

The time evolution of a beam of neutral boson system and of its decay products can be described as  $|\psi(t)\rangle = \psi_{M^0}(t)|M^0\rangle + \psi_{\bar{M}^0}(t)|\bar{M}^0\rangle + \sum_n d_n(t)|n\rangle$ , where  $M^0$  denotes  $K^0$ ,  $B_d^0$ ,  $B_s^0$  or  $D^0$ ;  $\bar{M}^0$  the corresponding antiparticles,  $|n\rangle$  are the decay products,  $t$  is the proper time,  $\psi_{M^0}(t)$ ,  $\psi_{\bar{M}^0}(t)$  and  $d_n(t)$  are time dependent functions. Since the decay products are absent at the instant of the  $M^0$  and  $\bar{M}^0$  production, the state vector at initial time  $t = 0$  is given by  $|\psi(0)\rangle = \psi_{M^0}(0)|M^0\rangle + \psi_{\bar{M}^0}(0)|\bar{M}^0\rangle$ .

If one is interested in evaluating only the wave functions  $\psi_{M^0}(t)$  and  $\psi_{\bar{M}^0}(t)$  and the times considered are much larger than the typical time scale of the strong interaction, then the time evolution of  $|\psi(t)\rangle$  can be well described, in the space formed by  $|M^0\rangle$  and  $|\bar{M}^0\rangle$ , by the Wigner-Weisskopf approximation [44]. The time evolution is thus determined by the Schrodinger equation  $i\frac{d}{dt}\Psi = \mathcal{H}\Psi$ , where  $\Psi = (\psi_{M^0}(t), \psi_{\bar{M}^0}(t))^T$  and the effective Hamiltonian  $\mathcal{H} = \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} \\ \mathcal{H}_{21} & \mathcal{H}_{22} \end{pmatrix}$  of the system is non-Hermitian. It can be written as  $\mathcal{H} = M - i\frac{\Gamma}{2}$  with  $M$  and  $\Gamma$  Hermitian matrices.

The matrix elements of  $\mathcal{H}$  are constrained by the conservation of discrete symmetries [4]:  $CPT$  conservation implies  $\mathcal{H}_{11} = \mathcal{H}_{22}$ ,  $T$  conservation entails  $|\mathcal{H}_{12}| = |\mathcal{H}_{21}|$  and  $CP$  conservation requires  $\mathcal{H}_{11} = \mathcal{H}_{22}$  and  $|\mathcal{H}_{12}| = |\mathcal{H}_{21}|$ .

Notice that in the presence of  $CP$  violation, i.e. for  $|\mathcal{H}_{12}| \neq |\mathcal{H}_{21}|$ , the mass matrix  $M$  and the decay matrix  $\Gamma$  do not commute,  $[M, \Gamma] \neq 0$ , then the Hamiltonian  $\mathcal{H}$  is non-Hermitian,  $\mathcal{H} \neq \mathcal{H}^\dagger$  and non-normal,  $[\mathcal{H}, \mathcal{H}^\dagger] \neq 0$ . In this case, the left and right eigenstates of  $\mathcal{H}$  are independent sets of vectors that are not connected by complex conjugation. This implies that  $\mathcal{H}$  cannot be diagonalized by a single unitary transformation but one has to use the rules of non-Hermitian quantum mechanics. In the following, the biorthonormal basis formalism [45] - [48] will be used and the discussion presented in Ref.[48] will be applied to describe the time evolution of mixed mesons in the presence of  $CP$  violation (see also Ref.[49]).

Let  $\lambda_j = m_j - i\Gamma_j/2$ , with  $j = L, H$  ( $L$  denotes the light mass state and  $H$  the heavy mass state<sup>1</sup>), be the eigenvalues of the Hamiltonian  $\mathcal{H}$ , with  $|M_j\rangle$  the corresponding eigenvectors:

$$\mathcal{H}|M_j\rangle = \lambda_j|M_j\rangle. \quad (1)$$

Denoting with  $\varepsilon_j$  and  $|\widetilde{M}_j\rangle$ , ( $j = L, H$ ) the eigenvalues and the eigenvectors of  $\mathcal{H}^\dagger$ :  $\mathcal{H}^\dagger|\widetilde{M}_j\rangle = \varepsilon_j|\widetilde{M}_j\rangle$ ; this equation can be recast in the form

$$|\widetilde{M}_j\rangle\mathcal{H} = \langle\widetilde{M}_j|\varepsilon_j^*. \quad (2)$$

By projecting Eq.(2) on the state  $|M_j\rangle$  one has  $\langle\widetilde{M}_j|\mathcal{H}|M_j\rangle = \langle\widetilde{M}_j|\varepsilon_j^*|M_j\rangle = \langle\widetilde{M}_j|\lambda_j|M_j\rangle$ , then  $\varepsilon_j^* = \lambda_j$ , i.e. the eigenvalues of  $\mathcal{H}$  are the complex conjugates of those of  $\mathcal{H}^\dagger$ . Moreover one has  $\langle\widetilde{M}_i|\mathcal{H}|M_j\rangle = \langle\widetilde{M}_i|\varepsilon_i^*|M_j\rangle = \langle\widetilde{M}_i|\lambda_j|M_j\rangle$ , hence:  $(\lambda_j - \varepsilon_i^*)\langle\widetilde{M}_i|M_j\rangle = 0$ . This last relation together with  $\lambda_j \neq \varepsilon_i^*$  for  $i \neq j$  implies the biorthogonality relation

$$\langle\widetilde{M}_i|M_j\rangle = \langle M_j|\widetilde{M}_i\rangle = \delta_{ij}. \quad (3)$$

Let us now derive the completeness relation. The state vector  $|\psi(t)\rangle$  of the neutral boson system (without its decay products) can be expressed as  $|\psi(t)\rangle = \sum_{j=1,2} a_j(t)|M_j\rangle = \sum_{j=1,2} \widetilde{a}_j(t)|\widetilde{M}_j\rangle$ , with  $a_j(t) = \langle\widetilde{M}_j|\psi(t)\rangle$  and  $\widetilde{a}_j(t) = \langle M_j|\psi(t)\rangle$ , i.e.  $|\psi(t)\rangle = \sum_{j=1,2} |M_j\rangle\langle\widetilde{M}_j|\psi(t)\rangle = \sum_{j=1,2} |\widetilde{M}_j\rangle\langle M_j|\psi(t)\rangle$ . This last equation implies the completeness relations

$$\sum_j |M_j\rangle\langle\widetilde{M}_j| = \sum_j |\widetilde{M}_j\rangle\langle M_j| = 1. \quad (4)$$

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<sup>1</sup> For  $K$  mesons, usually the mass eigenstates are defined according to their lifetimes:  $K_S$  is the short lived and  $K_L$  is the long lived; in this system  $K_L$  is the heavier state.

Summarizing, since in the presence of  $CP$  violation the effective Hamiltonian  $\mathcal{H}$  of mixed meson systems is non-Hermitian and non-normal, then the conjugate states  $\langle \widetilde{M}_j |^\dagger \equiv |\widetilde{M}_j\rangle$  and  $|M_j\rangle^\dagger \equiv \langle M_j|$  are not isomorphic to their duals:  $|\widetilde{M}_j\rangle \neq |M_j\rangle$  and  $\langle M_j| \neq \langle \widetilde{M}_j|$ . In this case, as a consequence of Eqs.(3) and (4), the set of states  $\{|M_j\rangle, \langle \widetilde{M}_j|\}$ , with  $j = L, H$ , is a complete biorthonormal system for  $\mathcal{H}$ .<sup>2</sup>

Furthermore, since the time evolution operator associated with  $\mathcal{H}$ ,  $U(t) = e^{-i\mathcal{H}t}$  is not unitary, one also introduces the time evolution operator of  $\mathcal{H}^\dagger$ ,  $\widetilde{U}(t) = e^{-i\mathcal{H}^\dagger t}$ , which satisfies  $U\widetilde{U}^\dagger = \widetilde{U}^\dagger U = 1$ . The spectral form of the Hamiltonian and of the operators  $U(t)$  and  $\widetilde{U}(t)$  are then given by

$$\mathcal{H} = \sum_{j=L,H} \lambda_j |M_j\rangle \langle \widetilde{M}_j|, \quad U(t) = \sum_{j=L,H} e^{-i\lambda_j t} |M_j\rangle \langle \widetilde{M}_j|, \quad \widetilde{U}(t) = \sum_{j=L,H} e^{-i\lambda_j^* t} |\widetilde{M}_j\rangle \langle M_j|, \quad (5)$$

respectively. Thus the time evolved of the states  $|M_k\rangle$  and  $|\widetilde{M}_k\rangle$ , ( $k = L, H$ ) at time  $t$  are  $|M_k(t)\rangle = U(t)|M_k\rangle = e^{-i\lambda_k t}|M_k\rangle$  and  $|\widetilde{M}_k(t)\rangle = \widetilde{U}(t)|\widetilde{M}_k\rangle = e^{-i\lambda_k^* t}|\widetilde{M}_k\rangle$  and the corresponding conjugate states are  $\langle M_k(t)| = \langle M_k|U^\dagger(t) = \langle M_k|e^{i\lambda_k^* t}$  and  $\langle \widetilde{M}_k(t)| = \langle \widetilde{M}_k|\widetilde{U}^\dagger(t) = \langle \widetilde{M}_k|e^{i\lambda_k t}$ .

Note that the existence of a complete biorthonormal set of eigenvector of  $\mathcal{H}$  implies that  $\mathcal{H}$  is diagonalizable. Thus, a matrix  $V$  exists such that  $V^{-1}HV = \text{diag}(\lambda_L, \lambda_H)$ , with

$$V = \begin{pmatrix} p_L & p_H \\ q_L & -q_H \end{pmatrix}, \quad V^{-1} = \frac{1}{q_L p_H + q_H p_L} \begin{pmatrix} q_H & p_H \\ q_L & -p_L \end{pmatrix}, \quad (6)$$

where  $q_L = c_L$ ,  $p_L = c_L \left( \frac{\lambda_L - \mathcal{H}_{22}}{\mathcal{H}_{21}} \right)$ ,  $q_H = -c_H$ ,  $p_H = c_H \left( \frac{\lambda_H - \mathcal{H}_{22}}{\mathcal{H}_{21}} \right)$  and  $c_L, c_H$  are complex constants:  $c_L, c_H \in \mathbb{C} - \{0\}$ . The right and left eigenvectors of the Hamiltonian  $\mathcal{H}$  are  $(p_L, q_L)^T$ ,  $(p_H, -q_H)^T$  and  $\frac{1}{q_L p_H + q_H p_L} (q_H, p_H)$ ,  $\frac{1}{q_L p_H + q_H p_L} (q_L, -p_L)$  respectively.<sup>3</sup> The explicit expressions of the mass eigenstates  $|M_L\rangle$ ,  $|M_H\rangle$ ,  $\langle \widetilde{M}_L|$  and  $\langle \widetilde{M}_H|$  in terms of the parameters  $q_j$  and  $p_j$ ,  $j = L, H$  are given in Appendix A. It is now convenient to introduce the  $CP$  and  $CPT$  parameters and to express the meson states in terms of these parameters. The constraints on  $\mathcal{H}$  imposed by  $CP$  and  $T$  invariance suggest to adopt the following  $CP$  and  $T$  violation parameter:

$$\varepsilon = \frac{|p_H/q_H| - |q_L/p_L|}{|p_H/q_H| + |q_L/p_L|} = \frac{|p/q| - |q/p|}{|p/q| + |q/p|} = \frac{|\mathcal{H}_{12}| - |\mathcal{H}_{21}|}{|\mathcal{H}_{12}| + |\mathcal{H}_{21}|}, \quad (7)$$

where

$$\frac{q}{p} = \sqrt{\frac{q_L q_H}{p_L p_H}} = \sqrt{\frac{\mathcal{H}_{21}}{\mathcal{H}_{12}}}. \quad (8)$$

Moreover,  $CPT$  invariance imposes the equality of the diagonal elements of the Hamiltonian  $\mathcal{H}$ ,  $\mathcal{H}_{11} = \mathcal{H}_{22}$ . Thus such an invariance can be tested by checking that the difference  $\mathcal{H}_{22} - \mathcal{H}_{11}$  is equal to zero.  $CPT$  violation can be described conveniently by the quantity  $z$  which is independent of phase convention

$$z = \frac{\frac{q_L}{p_L} - \frac{q_H}{p_H}}{\frac{q_L}{p_L} + \frac{q_H}{p_H}} = \frac{(\mathcal{H}_{22} - \mathcal{H}_{11})}{\lambda_L - \lambda_H}. \quad (9)$$

Notice that in the standard model extension (SME) the parameter  $z$  depends on the four-momentum of the meson [50], moreover, in the case of  $CPT$  invariance:  $p/q = p_L/q_L = p_H/q_H$  and  $z = 0$ .

By using Eqs.(8) and (9), the mass eigenstates  $|M_L\rangle$  and  $|M_H\rangle$  can be written in terms of  $|M^0\rangle$  and  $|\bar{M}^0\rangle$  as

$$|M_L\rangle = p\sqrt{1-z}|M^0\rangle + q\sqrt{1+z}|\bar{M}^0\rangle, \quad (10)$$

$$|M_H\rangle = p\sqrt{1+z}|M^0\rangle - q\sqrt{1-z}|\bar{M}^0\rangle, \quad (11)$$

<sup>2</sup> Note that Eqs.(1), (2), (3) and (4) do not determine the biorthonormal system  $\{|M_j\rangle, |\widetilde{M}_j\rangle\}$  uniquely. Any other biorthonormal system,  $\{|M'_j\rangle, |\widetilde{M}'_j\rangle\}$ , satisfying these conditions has the form  $|M'_j\rangle = \alpha_j |M_j\rangle$  and  $|\widetilde{M}'_j\rangle = \frac{1}{\alpha_j^*} |\widetilde{M}_j\rangle$ , with  $\alpha_j$  complex. This fact however does not affect any measurable quantity [49].

<sup>3</sup> If we impose the normalization conditions of  $|M_j\rangle$ ,  $j = 1, 2$ , we have  $|c_L| = \frac{|\mathcal{H}_{21}|}{\sqrt{|\lambda_L - \mathcal{H}_{22}|^2 + |\mathcal{H}_{21}|^2}}$ ,  $|c_H| = \frac{|\mathcal{H}_{21}|}{\sqrt{|\mathcal{H}_{22} - \lambda_H|^2 + |\mathcal{H}_{21}|^2}}$  and  $|p_L|^2 + |q_L|^2 = |p_H|^2 + |q_H|^2 = 1$ .

and, in a similar way,  $\langle \widetilde{M}_L |$  and  $\langle \widetilde{M}_H |$  are expressed as

$$\langle \widetilde{M}_L | = \frac{1}{2pq} \left[ q \sqrt{1-z} \langle \widetilde{M}^0 | + p \sqrt{1+z} \langle \widetilde{M}^0 | \right], \quad (12)$$

$$\langle \widetilde{M}_H | = \frac{1}{2pq} \left[ q \sqrt{1+z} \langle \widetilde{M}^0 | - p \sqrt{1-z} \langle \widetilde{M}^0 | \right]. \quad (13)$$

Thus, at time  $t$ , the states  $|M^0(t)\rangle$  and  $|\bar{M}^0(t)\rangle$  in terms of  $|M_L\rangle$  and  $|M_H\rangle$  are

$$|M^0(t)\rangle = \frac{1}{2p} \left[ \sqrt{1-z} |M_L\rangle e^{-i\lambda_L t} + \sqrt{1+z} |M_H\rangle e^{-i\lambda_H t} \right], \quad (14)$$

$$|\bar{M}^0(t)\rangle = \frac{1}{2q} \left[ \sqrt{1+z} |M_L\rangle e^{-i\lambda_L t} - \sqrt{1-z} |M_H\rangle e^{-i\lambda_H t} \right], \quad (15)$$

$$\langle \widetilde{M}^0(t) | = p \left[ \sqrt{1-z} \langle \widetilde{M}_L | e^{i\lambda_L t} + \sqrt{1+z} \langle \widetilde{M}_H | e^{i\lambda_H t} \right], \quad (16)$$

$$\langle \widetilde{M}^0(t) | = q \left[ \sqrt{1+z} \langle \widetilde{M}_L | e^{i\lambda_L t} - \sqrt{1-z} \langle \widetilde{M}_H | e^{i\lambda_H t} \right]. \quad (17)$$

The states in Eqs.(14)-(17) are the correct ones to be used in computations.

### III. MESON MIXING AND MUKUNDA-SIMON PHASE

The main result of the paper is presented in this Section; the Mukunda-Simon phases appearing in the time evolution of mixed meson  $M^0 - \bar{M}^0$  systems are related to the parameter describing the  $CPT$  violation and a difference between the non-cyclic phases of particles and of antiparticles signals  $CPT$  symmetry breaking. Moreover, the non-cyclic phases due to the particle-antiparticle oscillations are related to the parameters denoting the  $CP$  violation. A system particularly appropriate to study such phases is the  $B_s^0 - \bar{B}_s^0$  one.

Let us start by introducing the Mukunda-Simon phases for Hermitian systems. Subsequently, we consider such phases in the case of non-Hermitian systems and of mixed mesons.

Consider a quantum system whose state vector  $|\psi(t)\rangle$  evolves according the Schrodinger equation  $i(d/dt)|\psi(t)\rangle = H(t)|\psi(t)\rangle$ ; the Mukunda-Simon phase is defined as [24]:  $\Phi(t) = \arg\langle\psi(0)|\psi(t)\rangle - \Im \int_0^t \langle\dot{\psi}(t')|\dot{\psi}(t')\rangle dt'$ , where the dot denotes the derivative with respect to  $t'$ . The generalization of the above phase to the case of a system with a diagonalizable non-Hermitian Hamiltonian  $\mathcal{H}_{NH}(t)$  is presented in Ref.[24]. Its extension to the biorthonormal basis formalism is the following. Denoting with  $|\psi_{NH}(t)\rangle$  and  $|\tilde{\psi}_{NH}(t)\rangle$  the solution to the Schrodinger equation  $i(d/dt)|\psi_{NH}(t)\rangle = \mathcal{H}_{NH}(t)|\psi_{NH}(t)\rangle$  and to its adjoint equation  $i(d/dt)|\tilde{\psi}_{NH}(t)\rangle = \mathcal{H}_{NH}^\dagger(t)|\tilde{\psi}_{NH}(t)\rangle$ , respectively, the Mukunda-Simon phase is given by:  $\Phi_{NH}(t) = \arg\langle\tilde{\psi}_{NH}(0)|\psi_{NH}(t)\rangle - \Im \int_0^t \langle\tilde{\psi}_{NH}(t')|\dot{\psi}_{NH}(t')\rangle dt'$ . Such a quantity is reparametrization invariant and it is invariant under the complex gauge transformations  $|\psi_{NH}(t)\rangle \rightarrow |\psi'_{NH}(t)\rangle = S^{-1}(t)|\psi_{NH}(t)\rangle$  and  $|\tilde{\psi}_{NH}(t)\rangle \rightarrow |\tilde{\psi}'_{NH}(t)\rangle = W^{-1}(t)|\tilde{\psi}_{NH}(t)\rangle$ , where  $W(t) = [S^{-1}(t)]^\dagger$ . Therefore,  $\Phi_{NH}(t)$  is a geometric phase associated with the evolution of a quantum non-Hermitian system.

In the particular case of mixed meson systems  $M^0 - \bar{M}^0$  one has the following phases:

$$\Phi_{M^0 M^0}(t) = \arg\langle\widetilde{M}^0(0)|M^0(t)\rangle - \Im \int_0^t \langle\widetilde{M}^0(t')|\dot{M}^0(t')\rangle dt', \quad (18)$$

$$\Phi_{\bar{M}^0 \bar{M}^0}(t) = \arg\langle\widetilde{M}^0(0)|\bar{M}^0(t)\rangle - \Im \int_0^t \langle\widetilde{M}^0(t')|\dot{\bar{M}}^0(t')\rangle dt', \quad (19)$$

$$\Phi_{M^0 \bar{M}^0}(t) = \arg\langle\widetilde{M}^0(0)|\bar{M}^0(t)\rangle - \Im \int_0^t \langle\widetilde{M}^0(t')|\dot{\bar{M}}^0(t')\rangle dt', \quad (20)$$

$$\Phi_{\bar{M}^0 M^0}(t) = \arg\langle\widetilde{M}^0(0)|M^0(t)\rangle - \Im \int_0^t \langle\widetilde{M}^0(t')|\dot{M}^0(t')\rangle dt'. \quad (21)$$

$\Phi_{M^0 M^0}(t)$  and  $\Phi_{\bar{M}^0 \bar{M}^0}(t)$  are the phases of the particle  $M^0$  and of the antiparticle  $\bar{M}^0$ , respectively, and they are connected to  $CPT$  violation parameter;  $\Phi_{M^0 \bar{M}^0}(t)$  and  $\Phi_{\bar{M}^0 M^0}(t)$  are the phases due to particle-antiparticle oscillations, and they are linked to  $CP$  violation (see below).

Let us analyze in more detail such phases by starting with  $\Phi_{M^0 M^0}(t)$  and  $\Phi_{\bar{M}^0 \bar{M}^0}(t)$ . By using Eqs.(14)-(17), their explicit form is given by

$$\begin{aligned} \Phi_{M^0 M^0}(t) = & \arg \left[ e^{\frac{\Delta\Gamma t}{4}} [(1 - \Re z) \cos(m_L t) - \Im z \sin(m_L t)] + e^{-\frac{\Delta\Gamma t}{4}} [(1 + \Re z) \cos(m_H t) + \Im z \sin(m_H t)] \right. \\ & - i \left[ e^{\frac{\Delta\Gamma t}{4}} [(1 - \Re z) \sin(m_L t)] + \Im z \cos(m_L t) \right] + e^{-\frac{\Delta\Gamma t}{4}} [(1 + \Re z) \sin(m_H t) - \Im z \cos(m_H t)] \left. \right] \\ & + \frac{t}{2} \left( m + \Delta m \Re z + \frac{\Delta\Gamma}{2} \Im z \right), \end{aligned} \quad (22)$$

and

$$\begin{aligned} \Phi_{\bar{M}^0 \bar{M}^0}(t) = & \arg \left[ e^{\frac{\Delta\Gamma t}{4}} [(1 + \Re z) \cos(m_L t) + \Im z \sin(m_L t)] + e^{-\frac{\Delta\Gamma t}{4}} [(1 - \Re z) \cos(m_H t) - \Im z \sin(m_H t)] \right. \\ & - i \left[ e^{\frac{\Delta\Gamma t}{4}} [(1 + \Re z) \sin(m_L t) - \Im z \cos(m_L t)] + e^{-\frac{\Delta\Gamma t}{4}} [(1 - \Re z) \sin(m_H t) + \Im z \cos(m_H t)] \right] \left. \right] \\ & + \frac{t}{2} \left( m - \Delta m \Re z - \frac{\Delta\Gamma}{2} \Im z \right), \end{aligned} \quad (23)$$

respectively, where  $m = m_L + m_H$ ,  $\Delta m = m_H - m_L$  and  $\Delta\Gamma = \Gamma_H - \Gamma_L$ .<sup>4</sup> The symbol  $\Gamma$  denotes  $\Gamma = \Gamma_L + \Gamma_H$  (Appendix B). Assuming  $\frac{\Delta\Gamma t}{2}$  small, which is valid in the range  $|\Delta t| < 15ps$  used in the experimental analysis on  $B^0 - \bar{B}^0$  system [51], [53], Eqs.(22) and (23) become

$$\Phi_{M^0 M^0}(t) \simeq \arg \left[ \cos \left( \frac{\Delta m t}{2} \right) + (\Im z - i \Re z) \sin \left( \frac{\Delta m t}{2} \right) \right] + \frac{t}{2} \left( \Delta m \Re z + \frac{\Delta\Gamma}{2} \Im z \right), \quad (24)$$

and

$$\Phi_{\bar{M}^0 \bar{M}^0}(t) \simeq \arg \left[ \cos \left( \frac{\Delta m t}{2} \right) - (\Im z - i \Re z) \sin \left( \frac{\Delta m t}{2} \right) \right] - \frac{t}{2} \left( \Delta m \Re z + \frac{\Delta\Gamma}{2} \Im z \right), \quad (25)$$

respectively. These equations show the dependence of the phases on the real and imaginary part of the  $z$  parameter defined in Eq.(9). In particular, the difference between  $\Phi_{M^0 M^0}(t)$  and  $\Phi_{\bar{M}^0 \bar{M}^0}(t)$ :  $\Delta\Phi(t) = \Phi_{M^0 M^0}(t) - \Phi_{\bar{M}^0 \bar{M}^0}(t)$  is due to terms related to  $z$  and it is non-zero only in the presence of  $CPT$  violation. Indeed, in the case of  $CPT$  invariance,  $z = 0$ , one has  $\Phi_{M^0 M^0}^{CPT}(t) = \Phi_{\bar{M}^0 \bar{M}^0}^{CPT}(t) = \arg \left[ \cos \left( \frac{\Delta m t}{2} \right) \right]$ , which is trivially equal to 0 or  $\pi$  and  $\Delta\Phi(t) = 0$ .

Coming back now to the phases  $\Phi_{M^0 \bar{M}^0}(t)$  and  $\Phi_{\bar{M}^0 M^0}(t)$ , their explicit expressions are

$$\Phi_{M^0 \bar{M}^0}(t) = \frac{\pi}{2} - \frac{m}{2}t + \arg \left[ \frac{p}{q} \sqrt{1 - z^2} \sin \left[ \left( \Delta m - \frac{i\Delta\Gamma}{2} \right) \frac{t}{2} \right] \right] + \Im \left[ i \frac{p}{q} \sqrt{1 - z^2} \left( \Delta m - \frac{i\Delta\Gamma}{2} \right) \frac{t}{2} \right], \quad (26)$$

and

$$\Phi_{\bar{M}^0 M^0}(t) = \frac{\pi}{2} - \frac{m}{2}t + \arg \left[ \frac{q}{p} \sqrt{1 - z^2} \sin \left[ \left( \Delta m - \frac{i\Delta\Gamma}{2} \right) \frac{t}{2} \right] \right] + \Im \left[ i \frac{q}{p} \sqrt{1 - z^2} \left( \Delta m - \frac{i\Delta\Gamma}{2} \right) \frac{t}{2} \right]. \quad (27)$$

For  $\frac{\Delta\Gamma t}{2} \ll 1$  and omitting second order terms in  $z$ , one obtains

$$\Phi_{M^0 \bar{M}^0}(t) = \frac{\pi}{2} - \frac{mt}{2} + \arg \left[ \frac{p}{q} \sin \left( \frac{\Delta m t}{2} \right) \right] - \frac{\Delta m t}{2} \Re \left( \frac{p}{q} \right) - \frac{\Delta\Gamma t}{2} \Im \left( \frac{p}{q} \right), \quad (28)$$

and

$$\Phi_{\bar{M}^0 M^0}(t) = \frac{\pi}{2} - \frac{mt}{2} + \arg \left[ \frac{q}{p} \sin \left( \frac{\Delta m t}{2} \right) \right] - \frac{\Delta m t}{2} \Re \left( \frac{q}{p} \right) - \frac{\Delta\Gamma t}{2} \Im \left( \frac{q}{p} \right), \quad (29)$$

and the phase difference is  $\Phi_{M^0 \bar{M}^0}(t) - \Phi_{\bar{M}^0 M^0}(t) \neq 0$ . On the contrary, in the case of  $CP$  conservation one has

$$\Phi_{M^0 \bar{M}^0}^{CP}(t) = \Phi_{\bar{M}^0 M^0}^{CP}(t) = \frac{\pi}{2} - (m + \Delta m) \frac{t}{2} + \arg \left[ \sin \left( \frac{\Delta m t}{2} \right) \right], \quad (30)$$

---

<sup>4</sup> The sign of  $\Delta\Gamma$  is not yet established for  $B$  and  $B_s$  mesons, while  $\Delta\Gamma < 0$  for  $K$  mesons and  $\Delta\Gamma > 0$  for  $D$  mesons.

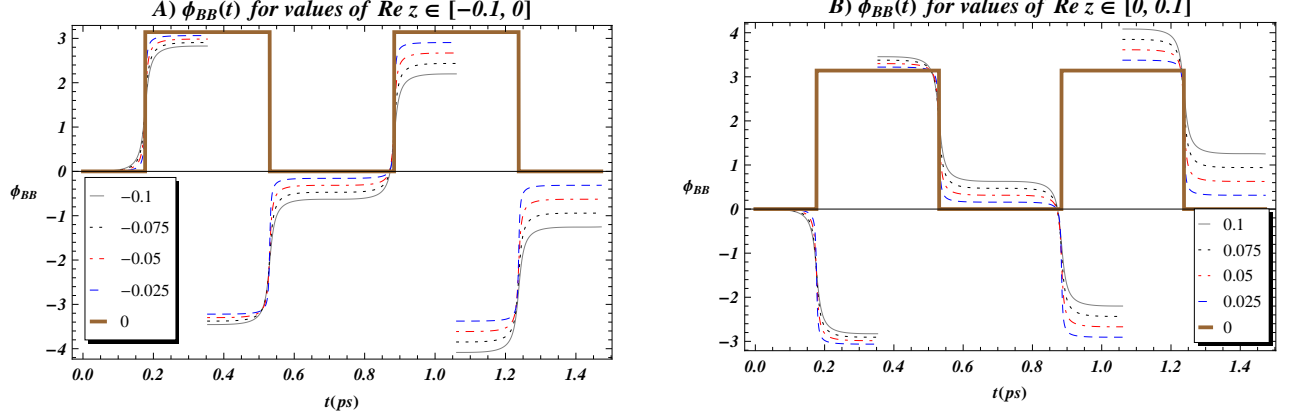


Figure 1: Plots of  $\Phi_{B_s^0 B_s^0}$  as a function of time  $t$  for  $\Im z = 0$  and different values of  $\Re z$ . In picture A)  $\Phi_{B_s^0 B_s^0}(t)$  is reported for sample values of  $\Re z \in [-0.1, 0]$  as indicated in the inset. In picture B)  $\Phi_{B_s^0 B_s^0}(t)$  is plotted for sample values of  $\Re z \in [0, 0.1]$  as shown in the inset.

and there is no phase difference.

*Numerical analysis:* The features of the phases related to the parameter  $z$ ,  $\Phi_{M^0 M^0}$ ,  $\Phi_{\bar{M}^0 \bar{M}^0}$  and  $\Delta\Phi$  are analyzed in detail for the  $B_s$  system. Such a system is particularly appropriate for the study of non-cyclic phases since many particle oscillations occur within its lifetime. Another useful system for such analysis is the neutral kaon one [54].

For the  $B_s$  mesons, one takes  $m_s = 1.63007 \times 10^{13} ps^{-1}$ ,  $\Delta m_s = 17.77 ps^{-1}$ ,  $\Gamma_s = 0.678 ps^{-1}$ ,  $\Delta\Gamma_s = -0.062 ps^{-1}$ . Moreover, one considers values of  $\Re z$  and  $\Im z$  in the intervals:  $-0.1 \leq \Re z \leq 0.1$  and  $-0.1 \leq \Im z \leq 0.1$  which are consistent with the experimental data [53]. Notice that, in such intervals for  $\Re z$  and  $\Im z$ , the phases  $\Phi_{B_s^0 B_s^0}$ ,  $\Phi_{\bar{B}_s^0 \bar{B}_s^0}$  and  $\Delta\Phi$  are weakly depending on the value of  $\Im z$ . Indeed, for example, in the time interval of the  $B_s^0$  life time, the shape variation of  $\Phi_{B_s^0 B_s^0}$  and  $\Phi_{\bar{B}_s^0 \bar{B}_s^0}$  with  $\Im z$  is at most of the order of 0.2%, so that one can fix an arbitrary value of  $\Im z$  in the values interval  $[-0.1, 0.1]$  and study the non-cyclic phases as functions of time for different values of  $\Re z$ . In Figs. (1), (2) and (3) the phases are drawn for  $\Im z = 0$ . In order to better show the behavior of the phases, the figures contain two plots A) and B) of the same phase for sample values of  $\Re z$  belonging to the intervals  $[-0.1, 0]$  and  $[0, 0.1]$ , respectively.

The plots show that in the case of  $CPT$  violation, the phases  $\Phi_{B_s^0 B_s^0}$  and  $\Phi_{\bar{B}_s^0 \bar{B}_s^0}$  are clearly non-trivial, in fact they can assume values different from 0 and  $\pi$  and, in particular, the phase difference  $\Delta\Phi$  is non-zero.

#### IV. CONCLUSIONS

In the presence of  $T$  violation the effective Hamiltonian of mixed meson systems is non-Hermitian and non-normal. The left and right eigenstates of  $\mathcal{H}$  are independent sets of vectors that are not connected by complex conjugation. Then  $\mathcal{H}$  cannot be diagonalized by a single unitary transformation but by a complete biorthonormal set of vectors.

The correct flavor states are then derived by using the biorthonormal basis formalism. They are used to compute the non-cyclic phases for oscillating mesons and the asymmetries describing the  $CP$  and  $CPT$  violations (see Appendix B). The obtained asymmetries are equivalent to the ones achieved by the usual formalism.

The main outcome of the present work is the study of the Mukunda Simon phases for mixed mesons and the discovery of the fact that the geometric phases appearing in the evolution of the meson  $\Phi_{M^0 M^0}(t)$  and of its antiparticle  $\Phi_{\bar{M}^0 \bar{M}^0}(t)$  depend on the  $CPT$  violating parameter  $z$ . In particular, only in the case of  $CPT$  symmetry breaking, such phases are non trivial and the phase difference  $\Delta\Phi$  between particle and antiparticle is non-zero.

The possibility of  $CPT$  violation has been investigated in detail by analyzing the Mukunda Simon phases for the neutral  $B_s$  system. Such a system, together with the kaon, is especially suitable for the study of geometric phases. The high precision of the upcoming experiments on the  $B_s^0$  mesons will allow us, in the next future, to completely determine the dynamics of such particles, thus such experiments and the ones analyzing kaons dynamics might allow

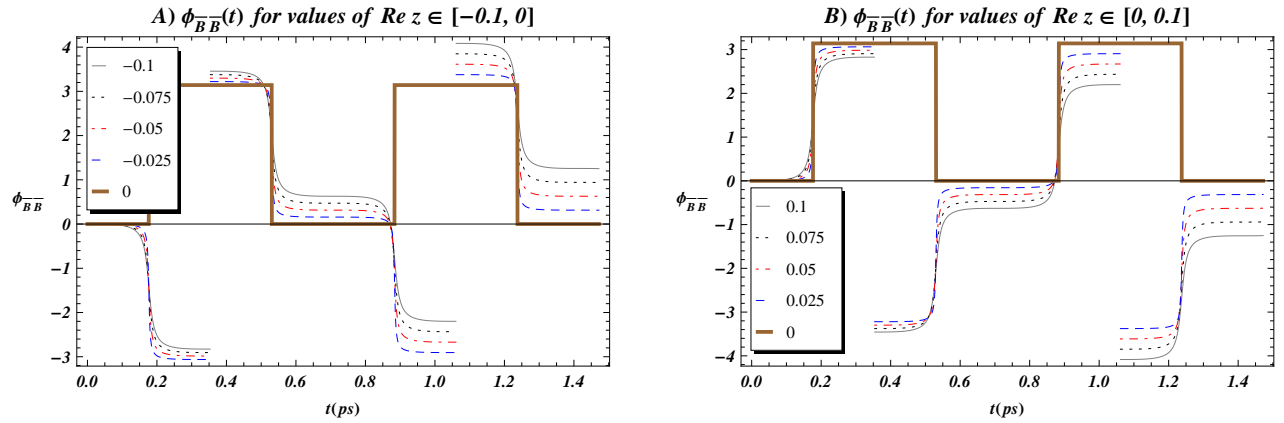


Figure 2: Plots of  $\Phi_{\bar{B}_s^0 \bar{B}_s^0}$  as a function of time for  $\Im z = 0$  and different sample values of  $\Re z$  as in Figure 1.

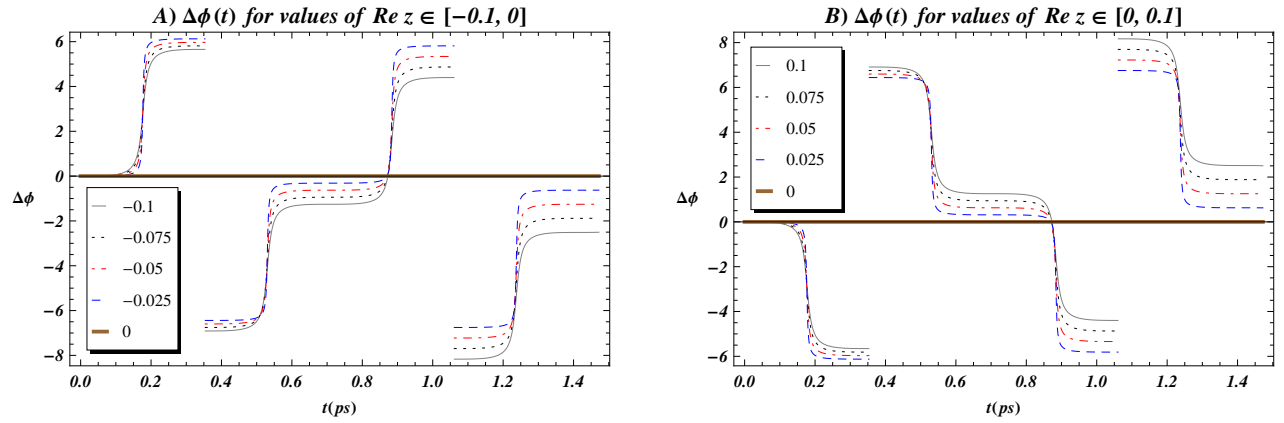


Figure 3: Plots of  $\Delta\Phi$  as a function of time for  $\Im z = 0$  and different sample values of  $\Re z$  as in Figures 1 and 2.

an accurate analysis of the geometric phases and in particular a measurement of the phase difference  $\Delta\Phi$  generated in the time evolution of the particle and the antiparticle. Such measurements might represent a completely alternative method to test one of the most important symmetries of the nature.

Finally, it has been also shown that the Mukunda-Simon phases  $\Phi_{M^0 \bar{M}^0}(t)$ ,  $\Phi_{\bar{M}^0 M^0}(t)$  and the Aharonov Anandan invariants  $s_{M^0 \bar{M}^0}(t)$ ,  $s_{\bar{M}^0 M^0}(t)$  (see Appendix C) due to meson oscillations are related to the  $CP$  violation parameters.  $CPT$  violation should not affect these phases, as the corrections are quadratic and expected to be negligible for small  $z$ . Thus, a study of the non-cyclic phases might be useful also for the analysis of the  $CP$  symmetry breaking.

### Acknowledgements

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### Appendix A: Other expressions of the states $|M^0(t)\rangle$ and $|\bar{M}^0(t)\rangle$

The mass eigenstates  $|M_L\rangle$  and  $|M_H\rangle$  are written in terms of  $|M^0\rangle$ ,  $|\bar{M}^0\rangle$  as <sup>5</sup>

$$|M_L\rangle = p_L |M^0\rangle + q_L |\bar{M}^0\rangle, \quad (\text{A1})$$

$$|M_H\rangle = p_H |M^0\rangle - q_H |\bar{M}^0\rangle, \quad (\text{A2})$$

and, in a similar way,  $\langle\widetilde{M}_L|$  and  $\langle\widetilde{M}_H|$  are expressed as

$$\langle\widetilde{M}_L| = \frac{1}{q_L p_H + q_H p_L} \left[ q_H \langle\widetilde{M}^0| + p_H \langle\widetilde{M}^0| \right], \quad (\text{A3})$$

$$\langle\widetilde{M}_H| = \frac{1}{q_L p_H + q_H p_L} \left[ q_L \langle\widetilde{M}^0| - p_L \langle\widetilde{M}^0| \right]. \quad (\text{A4})$$

Then at time  $t$ , the states  $|M^0(t)\rangle$  and  $|\bar{M}^0(t)\rangle$  in terms of  $|M_L\rangle$  and  $|M_H\rangle$  are:

$$|M^0(t)\rangle = \frac{1}{q_L p_H + q_H p_L} \left[ q_H |M_L\rangle e^{-i\lambda_L t} + q_L |M_H\rangle e^{-i\lambda_H t} \right], \quad (\text{A5})$$

$$|\bar{M}^0(t)\rangle = \frac{1}{q_L p_H + q_H p_L} \left[ p_H |M_L\rangle e^{-i\lambda_L t} - p_L |M_H\rangle e^{-i\lambda_H t} \right], \quad (\text{A6})$$

$$\langle\widetilde{M}^0(t)| = p_L \langle\widetilde{M}_L| e^{i\lambda_L t} + p_H \langle\widetilde{M}_H| e^{i\lambda_H t}, \quad (\text{A7})$$

$$\langle\widetilde{M}^0(t)| = q_L \langle\widetilde{M}_L| e^{i\lambda_L t} - q_H \langle\widetilde{M}_H| e^{i\lambda_H t}. \quad (\text{A8})$$

These states can be expressed also in the bases  $\{|M^0\rangle, |\bar{M}^0\rangle, \langle\widetilde{M}^0|, \langle\widetilde{M}^0|\}$  as

$$|M^0(t)\rangle = [g_+(t) + z g_-(t)] |M^0\rangle - \sqrt{1-z^2} \frac{q}{p} g_-(t) |\bar{M}^0\rangle, \quad (\text{A9})$$

$$|\bar{M}^0(t)\rangle = -\sqrt{1-z^2} \frac{p}{q} g_-(t) |M^0\rangle + [g_+(t) - z g_-(t)] |\bar{M}^0\rangle, \quad (\text{A10})$$

$$\langle\widetilde{M}^0(t)| = [\tilde{g}_+(t) + z \tilde{g}_-(t)] \langle\widetilde{M}^0| - \sqrt{1-z^2} \frac{p}{q} \tilde{g}_-(t) \langle\widetilde{M}^0|, \quad (\text{A11})$$

$$\langle\widetilde{M}^0(t)| = -\sqrt{1-z^2} \frac{q}{p} \tilde{g}_-(t) \langle\widetilde{M}^0| + [\tilde{g}_+(t) - z \tilde{g}_-(t)] \langle\widetilde{M}^0|, \quad (\text{A12})$$

with  $g_{\mp}(t) = \frac{1}{2}(e^{-i\lambda_H t} \mp e^{-i\lambda_L t})$  and  $\tilde{g}_{\mp}(t) = \frac{1}{2}(e^{i\lambda_H t} \mp e^{i\lambda_L t})$ .

### Appendix B: CP and CPT asymmetries

The expressions of the asymmetries  $A_T$  and  $A_{CPT}$  describing a departure from time reversal and  $CPT$  invariances, respectively, are calculated by using the states derived in the biorthonormal formalism, Eqs.(14)-(17). The obtained results are equivalent to the asymmetries computed by using the usual formalism [53], [55].

Let us begin by computing the  $T$  asymmetry. The violation of time reversal invariance can be revealed by the comparison between the probability of transition from  $\bar{M}^0$  to  $M^0$ ,  $P_{\bar{M}^0 \rightarrow M^0}$ , and the probability of transition from  $M^0$  to  $\bar{M}^0$ ,  $P_{M^0 \rightarrow \bar{M}^0}$ , in the asymmetry:

$$A_T(\Delta t) = \frac{P_{\bar{M}^0 \rightarrow M^0}(\Delta t) - P_{M^0 \rightarrow \bar{M}^0}(\Delta t)}{P_{\bar{M}^0 \rightarrow M^0}(\Delta t) + P_{M^0 \rightarrow \bar{M}^0}(\Delta t)} \quad (\text{B1})$$

with  $\Delta t = t_f - t_i$  denoting the time interval between the initial time  $t_i$  and the final time  $t_f$ . The transition amplitudes  $A_{\bar{M}^0 \rightarrow M^0}(\Delta t)$  and  $A_{M^0 \rightarrow \bar{M}^0}(\Delta t)$  are given respectively by

$$A_{\bar{M}^0 \rightarrow M^0}(\Delta t) = \langle\widetilde{M}^0(t_f)|\bar{M}^0(t_i)\rangle = \langle\widetilde{M}^0|e^{-iH\Delta t}|\bar{M}^0\rangle = \frac{1}{2} \frac{q}{p} \sqrt{1-z^2} (e^{-i\lambda_L \Delta t} - e^{-i\lambda_H \Delta t}), \quad (\text{B2})$$

$$A_{M^0 \rightarrow \bar{M}^0}(\Delta t) = \langle\widetilde{M}^0(t_f)|M^0(t_i)\rangle = \langle\widetilde{M}^0|e^{-iH\Delta t}|M^0\rangle = \frac{1}{2} \frac{p}{q} \sqrt{1-z^2} (e^{-i\lambda_L \Delta t} - e^{-i\lambda_H \Delta t}). \quad (\text{B3})$$

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<sup>5</sup> note that  $\langle M_j | M_{i \neq j} \rangle \neq 0$ ; for example, in the kaon case, one has  $\langle K_S | K_L \rangle \neq 0$ .



The results in Eqs.(B2)-(B3) are obtained by introducing the identity operator  $|M_L\rangle\langle\widetilde{M}_L| + |M_H\rangle\langle\widetilde{M}_H| = 1$  on the right side of the operator  $e^{-iH\Delta t}$ . The corresponding transition probabilities are then

$$P_{\bar{M}^0 \rightarrow M^0}(\Delta t) = \left| \langle \widetilde{M}^0(t_f) | \bar{M}^0(t_i) \rangle \right|^2 = \frac{1}{2} \left| \frac{q}{p} \right|^2 \left| \sqrt{1-z^2} \right|^2 e^{-\frac{\Gamma}{2}\Delta t} \left[ \cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right) - \cos(\Delta m\Delta t) \right], \quad (\text{B4})$$

$$P_{M^0 \rightarrow \bar{M}^0}(\Delta t) = \left| \langle \widetilde{M}^0(t_f) | M^0(t_i) \rangle \right|^2 = \frac{1}{2} \left| \frac{p}{q} \right|^2 \left| \sqrt{1-z^2} \right|^2 e^{-\frac{\Gamma}{2}\Delta t} \left[ \cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right) - \cos(\Delta m\Delta t) \right]. \quad (\text{B5})$$

The asymmetry  $A_T$  is time independent and it is given by

$$A_T = \frac{1 - \left| \frac{q}{p} \right|^4}{1 + \left| \frac{q}{p} \right|^4}. \quad (\text{B6})$$

A value different from zero of the quantity in Eq(B6) indicates a direct  $T$  violation independent from  $CPT$  violation. The result (B6) coincides with the Eq.(54) in Ref.[55].

In a similar way, the violation of  $CPT$  invariance can be revealed by the comparison between the probability of transition from  $M^0$  to  $M^0$ ,  $P_{M^0 \rightarrow M^0}$ , and the probability of transition from  $\bar{M}^0$  to  $\bar{M}^0$ ,  $P_{\bar{M}^0 \rightarrow \bar{M}^0}$ , in the asymmetry

$$A_{CPT}(\Delta t) = \frac{P_{M^0 \rightarrow M^0}(\Delta t) - P_{\bar{M}^0 \rightarrow \bar{M}^0}(\Delta t)}{P_{M^0 \rightarrow M^0}(\Delta t) + P_{\bar{M}^0 \rightarrow \bar{M}^0}(\Delta t)}. \quad (\text{B7})$$

The transition amplitudes  $A_{M^0 \rightarrow M^0}(\Delta t)$  and  $A_{\bar{M}^0 \rightarrow \bar{M}^0}(\Delta t)$  are given respectively by

$$A_{M^0 \rightarrow M^0}(\Delta t) = \langle \widetilde{M}^0(t_f) | M^0(t_i) \rangle = \langle \widetilde{M}^0 | e^{-iH\Delta t} | M^0 \rangle = \left( \frac{1+z}{2} \right) e^{-i\lambda_H\Delta t} + \left( \frac{1-z}{2} \right) e^{-i\lambda_L\Delta t}, \quad (\text{B8})$$

$$A_{\bar{M}^0 \rightarrow \bar{M}^0}(\Delta t) = \langle \widetilde{M}^0(t_f) | \bar{M}^0(t_i) \rangle = \langle \widetilde{M}^0 | e^{-iH\Delta t} | \bar{M}^0 \rangle = \left( \frac{1-z}{2} \right) e^{-i\lambda_H\Delta t} + \left( \frac{1+z}{2} \right) e^{-i\lambda_L\Delta t}, \quad (\text{B9})$$

where again the relation  $|M_L\rangle\langle\widetilde{M}_L| + |M_H\rangle\langle\widetilde{M}_H| = 1$  has been introduced on the right side of  $e^{-iH\Delta t}$ . The corresponding transition probabilities are then

$$P_{M^0 \rightarrow M^0}(\Delta t) = \left| \langle \widetilde{M}^0(t_f) | M^0(t_i) \rangle \right|^2 = e^{-\frac{\Gamma}{2}\Delta t} \left[ \left( \frac{1+|z|^2}{2} \right) \cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right) - \Re z \sinh\left(\frac{\Delta\Gamma\Delta t}{2}\right) + \left( \frac{1-|z|^2}{2} \right) \cos(\Delta m\Delta t) + \Im z \sin(\Delta m\Delta t) \right], \quad (\text{B10})$$

$$P_{\bar{M}^0 \rightarrow \bar{M}^0}(\Delta t) = \left| \langle \widetilde{M}^0(t_f) | \bar{M}^0(t_i) \rangle \right|^2 = e^{-\frac{\Gamma}{2}\Delta t} \left[ \left( \frac{1+|z|^2}{2} \right) \cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right) + \Re z \sinh\left(\frac{\Delta\Gamma\Delta t}{2}\right) + \left( \frac{1-|z|^2}{2} \right) \cos(\Delta m\Delta t) - \Im z \sin(\Delta m\Delta t) \right]. \quad (\text{B11})$$

The asymmetry  $A_{CPT}$  is thus given by

$$A_{CPT}(\Delta t) = \frac{-2 \Re z \sinh\left(\frac{\Delta\Gamma\Delta t}{2}\right) + 2 \Im z \sin(\Delta m\Delta t)}{(1+|z|^2) \cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right) + (1-|z|^2) \cos(\Delta m\Delta t)}. \quad (\text{B12})$$

Omitting second order terms in  $z$  and making the approximation  $\sinh\left(\frac{\Delta\Gamma\Delta t}{2}\right) \simeq \frac{\Delta\Gamma\Delta t}{2}$  which is valid in the range  $|\Delta t| < 15ps$  used in the experimental analysis of the  $B^0 - \bar{B}^0$  systems [52], [53], one has

$$A_{CPT}(\Delta t) \simeq \frac{-\Re z \Delta\Gamma\Delta t + 2 \Im z \sin(\Delta m\Delta t)}{\cosh\left(\frac{\Delta\Gamma\Delta t}{2}\right) + \cos(\Delta m\Delta t)}, \quad (\text{B13})$$

which coincides with Eq.(6) of Ref.[53]. In the case of  $CPT$  invariance,  $z = 0$  and  $A_{CPT} = 0$ .

### Appendix C: Aharonov–Anandan phase and $CP$ violation

The Aharonov–Anandan invariant is defined as [21]  $s = 2 \int_0^t \Delta E(t') dt'$ , where  $\Delta E$  is the variance of the energy  $E$ . The generalization to systems with a non-Hermitian Hamiltonian is presented in Ref.[46] where the biorthonormal basis formalism is also used. For a system with a complete biorthonormal basis  $\{|\psi(t)\rangle, \langle\tilde{\psi}(t)|\}$ , the variance is given by  $\Delta E_{NH}^2(t) = \langle\tilde{\psi}(t)|H^2|\psi(t)\rangle - (\langle\tilde{\psi}(t)|H|\psi(t)\rangle)^2$ , and the Aharonov–Anandan phase is  $s_{NH} = 2 \int_0^t |\Delta E_{NH}(t')| dt'$ . In the particular case of the mixed meson systems, one has the following variances:

$$\Delta E_{M^0 M^0}(t) = \Delta E_{\bar{M}^0 \bar{M}^0}(t) = \frac{1}{2} \sqrt{(1-z^2)} (\lambda_H - \lambda_L), \quad (C1)$$

$$\Delta E_{M^0 \bar{M}^0}(t) = \langle \widetilde{M^0}(t) | H | \bar{M}^0(t) \rangle = -\frac{p}{q} \frac{\sqrt{(1-z^2)}}{2} (\lambda_H - \lambda_L), \quad (C2)$$

$$\Delta E_{\bar{M}^0 M^0}(t) = \langle \widetilde{\bar{M}^0}(t) | H | M^0(t) \rangle = -\frac{q}{p} \frac{\sqrt{(1-z^2)}}{2} (\lambda_H - \lambda_L). \quad (C3)$$

Such relations show that the variances depend on  $z^2$ . Moreover, since  $\Delta E_{M^0 M^0}(t) = \Delta E_{\bar{M}^0 \bar{M}^0}(t)$ , then the corresponding invariants for particle and antiparticle are equal. These facts mean that Aharonov–Anandan invariants do not represent a good tool to test  $CPT$  invariance. However, such phases could be useful in the study of  $CP$  violation. Indeed, by neglecting the second order dependence on the  $z$  parameter, one has:

$$s_{M^0 M^0}(t) = s_{\bar{M}^0 \bar{M}^0}(t) = 2 \int_0^t |\Delta E_{M^0 M^0}(t')| dt' = \sqrt{(\Delta m)^2 + \frac{(\Delta \Gamma)^2}{4}} t, \quad (C4)$$

$$s_{M^0 \bar{M}^0}(t) = 2 \int_0^t |\Delta E_{M^0 \bar{M}^0}(t')| dt' = \left| \frac{p}{q} \right| \sqrt{(\Delta m)^2 + \frac{(\Delta \Gamma)^2}{4}} t, \quad (C5)$$

$$s_{\bar{M}^0 M^0}(t) = 2 \int_0^t |\Delta E_{\bar{M}^0 M^0}(t')| dt' = \left| \frac{q}{p} \right| \sqrt{(\Delta m)^2 + \frac{(\Delta \Gamma)^2}{4}} t. \quad (C6)$$

The phase  $s_{M^0 \bar{M}^0}(t)$  is different from  $s_{\bar{M}^0 M^0}(t)$  because of the  $CP$  violation  $p \neq q$ , independently from  $CPT$  violation. Eqs.(C5) and (C6) can be then used to compute the following quantity:

$$\frac{s_{M^0 \bar{M}^0} - s_{\bar{M}^0 M^0}}{s_{M^0 \bar{M}^0} + s_{\bar{M}^0 M^0}} = \frac{|p/q| - |q/p|}{|p/q| + |q/p|} = \frac{|H_{12}| - |H_{21}|}{|H_{12}| + |H_{21}|}, \quad (C7)$$

which coincides with the  $CP$  and  $T$  violating parameter  $\varepsilon$  defined in Eq.(7). Thus, the Aharonov–Anandan phases could represent a completely new way to estimate the parameter  $\varepsilon$  in mixed meson systems such as the  $K^0 - \bar{K}^0$  one [54]. In the case of  $CP$  conservation one should have  $s_{M^0 M^0}(t) = s_{\bar{M}^0 \bar{M}^0}(t) = s_{M^0 \bar{M}^0}(t) = s_{\bar{M}^0 M^0}(t)$  and  $\varepsilon = 0$ .

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